Math 210A Lecture 17 Notes

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1 Applications of the Sylow theorems

1.1 Groups of order p^n, pq, and p^2q

Proposition 1.1. *Groups of order p^n with $n > 1$ are not simple.*

Proof. Assume for contradiction that G is simple. Note that $Z(G) \| G\|$ and is nontrivial. So $Z(G) = G$, which makes G abelian. So G has order p. ☐

Proposition 1.2. *Groups of order pq with primes $p < q$ have a normal subgroup of order q and are cyclic if $q \not\equiv 1 \pmod p$.*

Proof. Note that $n_q(G) \mid p$, and $n_q(G) \equiv 1 \pmod q$. So $n_q(G) = 1$. By Sylow's theorem, $Q \trianglelefteq G$, where Q is a Sylow-q subgroup. So $PQ = G$, and $P \cap Q = \{e\}$, so $G = Q \rtimes P$. This gives a homomorphism $\varphi : P \to \mathrm{Aut}(Q)$. Moreover, $\mathrm{Aut}(Q) = (\mathbb{Z}.q\mathbb{Z})^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$. The map φ is trivial unless $q \cong 1 \pmod p$. If it is trivial, then $G = P \times Q = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$. ☐

Proposition 1.3. *Groups of order 255 are cyclic.*

Proof. Factor $255 = 3 \cdot 5 \cdot 17$. By the Sylow theorems, $n_17(G) = 1$, so we hav a normal Sylow 17-subgroup P such that $G/P \cong \mathbb{Z}/15\mathbb{Z}$. Look at $n_3(G)$ and $n_5(G)$. Note that $n_3(G) = 1$ or 85, and $n_5(G) = 1$ or 51. If $n_3(G) = 85$, we get $2 \cdot 85 = 170$ elements of order 3. If $n_5(G) = 51$, we have $4 \cdot 51 = 204$ elements of order 5. We cannot have both, so we either have a normal Sylow 3-subgroup or a normal Sylow 5-subgroup Q.

Then $PQ \trianglelefteq G$, and R is a Sylow-4 or Sylow-3 subgroup. Then $G = PQ \rtimes R$, with a homomorphism $R \to \mathrm{Aut}(PQ)$. Since PQ is cyclic, $\mathrm{Aut}(PQ) \cong \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Since R has order prime to 2, this homomorphism is trivial. So we get $G = P \times Q \times R \cong \mathbb{Z}/255\mathbb{Z}$. ☐

Proposition 1.4. *Groups of order p^2q with p, q prime are not simple.*

Proof. If $p > q$, then $n_p(G) \cong 1 \pmod{p}$ and $n_p(G) \mid q$, so $n_p(G) = 1$. If $q > p$. $n_q(G) = 1$ or p^2. Assume $n_q(G) = p^2$. Then $p^2 \cong 1 \pmod{q}$, so $q \mid (q-1)$ or $q \mid p+1$. Since $q > p$, we cannot have $q \mid (p-1)$, so we must have $q \mid (p+1)$, which gives $p = 2$ and $q = 3$. So $n_2(G) = 3$, and $n_q(G) = 4$. So there are 8 elements of order 3 and at least $3+2+1$ elements of 2-power order. But this gives 14 elements, which is greater than $12 = 2^2 \cdot 3$. $\square$

1.2 Subgroups of S_n

Proposition 1.5. *Suppose that G is finite, simple, and $p \mid |G|$ (but $p \not\|\, |G|$). Then G is isomorphic to a subgroup of S_n, where $n = n_p(G)$.*

Proof. G acts on $\mathrm{Syl}_p(G)$ by conjugation. There are n such Sylow p-subgroups, so this gives a homomorphism $\rho : G \to S_n$ such that $\ker(\rho) \trianglelefteq G$. If $\ker(\rho) = 1$, then G is isomorphic to a subgroup of S_n. If $\ker(G) = G$, the action is trivial but also transitive. So there exists a unique, therefore normal, Sylow p-subgroup. $\square$

Proposition 1.6. *There are no simple groups of order 160.*

Proof. Factor $160 = 2^5 \cdot 5$. If G is simple and $|G| = 160$, the $n_5(G) = 16$ and $n_2(G) = 5$. So G is isomorphic to a subgroup of S_5. But $|S_5| = 5! = 120$, which is a contradiction. $\square$

Proposition 1.7. *Let $H, K \leq G$ with H, K finite. Then $|HK| = |H||K|/|H \cap K|$.*

Proof. Consider the bijection $H/(H \cap K) \to HK/K$. Finish the rest for homework. $\square$

Proposition 1.8. *There are no simple groups of order 48.*

Proof. Factor $48 = 2^4 \cdot 3$. If G is simple, $n_2(G) = 3$. Let P, Q be Sylow 2-subgroups of G. Then $|P \cap Q| = |P||Q|/|PQ| = 256/|PQ|$. Since $|PQ > 48$, we get $|P \cap Q| > 4$. So $|P \cap Q| = 8$, which gives $|PQ| = 32$. Then $P \cap Q \trianglelefteq P, Q$. So $N_G(P \cap Q) \supseteq PQ$ must equal G, and we get that $P \cap Q \trianglelefteq G$. $\square$

This is a special case of the following proposition.

Proposition 1.9. *Let $p^n \mid\mid |G|$, and suppose that $|P \cap Q| \leq p^{n-r}$ for some $r \geq 1$ for all Sylow p subgroups $P \neq Q$. Then $n_p(G) \equiv 1 \pmod{p^r}$.*

Proof. The idea is to show that $P \cap Q = P \cap N_G(Q)$. We will do this next time. $\square$